SM3 9.5 Solving Trig Equations

Solving an equation where the variable is the argument of a trigonometric function is not much more complicated than solving one without trig, given sufficient unit circle skill or a calculator.

Step 1: Treat the trig function as though it were the variable being solved for and isolate it.

Step 2: Use your unit circle skills to remove the trig function. Realize that this will normally result in a set of values. Typically, you'll be solving on the domain of $[0,2\pi)$, so discard any other solutions. It is possible to have no solutions.

Step 3: Once the trig function is removed, if the variable is not isolated, continue to solve for the variable. We'll postpone needing this step until next time's lesson.

Keep in mind that we're solving an equation. This is not a request for proof. There is no need for the two-column format. I'll include reasons in the examples and answer key to clarify the process.

Example: Solve on $[0, 2\pi)$: $4\cos\theta + 14 = 12$
Strategy: The trig function, $\cos\theta$, must be isolated. We'll use the same algebraic steps we'd use to solve $4x + 14 = 12$ for x.

$$\begin{aligned} 4\cos\theta + 14 &= 12 & \\ 4\cos\theta &= -2 & \text{Subtract 14} \\ \cos\theta &= -\frac{1}{2} & \text{Divide by 4} \end{aligned}$$

Which values of θ will produce $\cos\theta = -\frac{1}{2}$? While it is somewhat helpful to use $\theta = \cos^{-1}\left(-\frac{1}{2}\right)$, the arc-trig functions only produce one solution. We are looking for every θ-value on $[0, 2\pi)$, not just the one that arccos happens to use.

$$\theta = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \qquad \text{Unit Circle}$$

We can check the solutions in the original equation to make certain we've been accurate. Some math checks are critical because of the existence of extraneous solutions. This happens with trig equations too, normally when both sides have been squared.

$$\begin{aligned} 4\cos\left(\frac{2\pi}{3}\right) + 14 &= 12 \\ 4\left(-\frac{1}{2}\right) + 14 &= 12 \\ -2 + 14 &= 12 \\ 12 &= 12 \end{aligned}$$

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Example: Solve on $[0, 2\pi)$: $8\sin^2\theta = 8$

Strategy: The trig function, $\sin\theta$, must be isolated. We'll use the same algebraic steps we'd use to solve $8x^2 = 8$ for x. It will be important to recall that taking the square root of both sides of an equations requires use of the $\pm$ symbol.

$$\begin{aligned} 8\sin^2\theta &= 8 \\ \sin^2\theta &= 1 && \text{Divide by 8} \\ \sin\theta &= \pm 1 && \text{Square root, remember your } \pm \\ \theta &= \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} && \text{Unit Circle} \end{aligned}$$

Example: Solve on $[0, 2\pi)$: $\sec^2\theta - 3\sec\theta + 7 = 5$

Strategy: The trig function, $\sec\theta$, must be isolated. But we have more than one copy of the trig function. Let's substitute $x = \sec\theta$ and see what we're dealing with.

$$x^2 - 3x + 7 = 5$$

This polynomial is a mere quadratic, we can use our quadratic-solving skills from Secondary Math 2 to help find the solution(s).

$$\begin{aligned} \sec^2\theta - 3\sec\theta + 7 &= 5 \\ \sec^2\theta - 3\sec\theta + 2 &= 0 && \text{Write the quadratic in } ax^2 + bx + c = 0 \text{ format} \\ (\sec\theta - 2)(\sec\theta - 1) &= 0 && \text{Factor} \\ \sec\theta - 2 = 0 \qquad \sec\theta - 1 &= 0 && \text{Set each factor} = 0 \text{ and solve.} \\ \sec\theta &= \{2, 1\} && \text{Addition} \\ \cos\theta &= \left\{\frac{1}{2}, 1\right\} && \text{No one likes } \sec\theta \\ \theta &= \left\{0, \frac{\pi}{3}, \frac{5\pi}{3}\right\} && \text{Unit Circle} \end{aligned}$$

Example: Solve on $[0, 2\pi)$: $1 + \cos\theta = \sin\theta$

Strategy: We've got more than one trig function. We don't have useful identities to work with unless the trig functions are squared, so let's square both sides to start.

$$\begin{aligned} 1 + \cos\theta &= \sin\theta \\ \cos^2\theta + 2\cos\theta + 1 &= \sin^2\theta && \text{Square both sides} \\ \cos^2\theta + 2\cos\theta + 1 &= 1 - \cos^2\theta && \text{Pyth ID} \\ 2\cos^2\theta + 2\cos\theta &= 0 && \text{Write the quadratic in } ax^2 + bx + c = 0 \text{ format} \\ 2\cos\theta\,(\cos\theta + 1) &= 0 && \text{Factor} \\ \cos\theta &= \{0, -1\} && \text{Solve both equations for } \cos\theta \\ \theta &= \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \pi\right\} && \text{Unit Circle} \end{aligned}$$

$$1 + \cos\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)$$
$$1 + 0 = 1$$

$$1 + \cos\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right)$$
$$1 + 0 \neq -1$$

$$1 + \cos(\pi) = \sin(\pi)$$
$$1 + 0 = 1$$

$$\boxed{\theta = \left\{\frac{\pi}{2}, \pi\right\}}$$

HW10.5

Solve each equation over the interval $[0, 2\pi)$.

1) $2\cos\theta + 4 = 5$

2) $2\sin\theta - 1 = 0$

3) $\tan^2\theta - 3 = 0$

4) $5\cos\theta - \sqrt{3} = 3\cos\theta$

5) $4\sec^2\theta - 2 = 0$

6) $\sin^2\theta - 5\cos\theta = 5$

7) $4\sin^2\theta - 2 = 0$

8) $3\tan\theta - \sqrt{3} = 0$

9) $\sec(\theta) + 2 = 0$

10) $\sin^2 \theta - 4 \sin \theta = 5$

11) $\cot \theta \sec \theta + \cot \theta = 0$

12) $5 \cos 2\theta + 1 = 3 \cos 2\theta$

Solve each equation over the interval $(-2\pi, 2\pi)$.

13) $16 \cos^2 \theta - 8 = 0$

14) $2 \cos^2 \theta - 3 \cos \theta + 1 = 0$